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Multifurcations at the origin of Zaslavsky's map with twist

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Abstract. As the nonlinear parameter, k of Zaslavsky's map with twist increases, chains of periodic points are born simultaneously at the origin and move outwards. The positions of the periodic points as a function of k is investigated, and in a limited number of cases an analytic result is found. When k is very small, it is found that there is a universal relation for the radius of the periodic points as a function of the nonlinear parameter.

1. Introduction

It has been well established that insight into the behaviour of dynamical systems may be obtained by studying the properties of maps derived from the equations of motion. One way to derive a map is to record dynamical variables at (well separated) equal time intervals, equivalent to examining it under a stroboscopic lamp. An alternative way is to use the technique of the Poincaré section, in which data is recorded whenever some dynamical variable has a predetermined value.

Many of the maps studied involve only one or two dynamical variables, e.g the one-dimensional logistic map and the two-dimensional standard map and Zaslavsky's map with twist. It is this last map, sometimes referred to as the stochastic web map, that we shall discuss in this paper.

Zaslavsky's map with twist, $\mathbf{M}_n$ is defined by

$$\mathbf{M}_n(k): \quad \begin{aligned} x_{j+1} &= (x_j + k\sin(y_j))\cos(2\pi/n) + y_j\sin(2\pi/n) \\ x_{j+1} &= -(x_j + k\sin(y_j))\sin(2\pi/n) + y_j\cos(2\pi/n). \end{aligned} \tag{1}$$

k is called the nonlinear parameter. The properties of this map, hereinafter referred to as the map with twist, are most interesting at resonance when the rotation number n is an integer: usually a small integer: 3, 4, 5 or 6. Some trajectories of this map are shown in figures 1 and 2.

The map with twist $\mathbf{M}_n$ has been extensively studied (Chernikov *et al* 1987, Borcherds and McCauley 1991, Zaslavsky *et al* 1991). It describes the dynamics in velocity space of a charged particle in a uniform steady magnetic field, subjected to an orthogonal electric field wavepacket. The harmonic components of the wavepacket have equal amplitude, and equal frequency spacing. How the map equations (1) are derived from the physical model is discussed in some detail by Zaslavsky *et al* (1986).

The properties of a map are conveniently studied through the behaviour of its fixed points and periodic points: those points which are mapped into themselves after a finite number of iterations of the mapping. (A fixed point is a special case of a

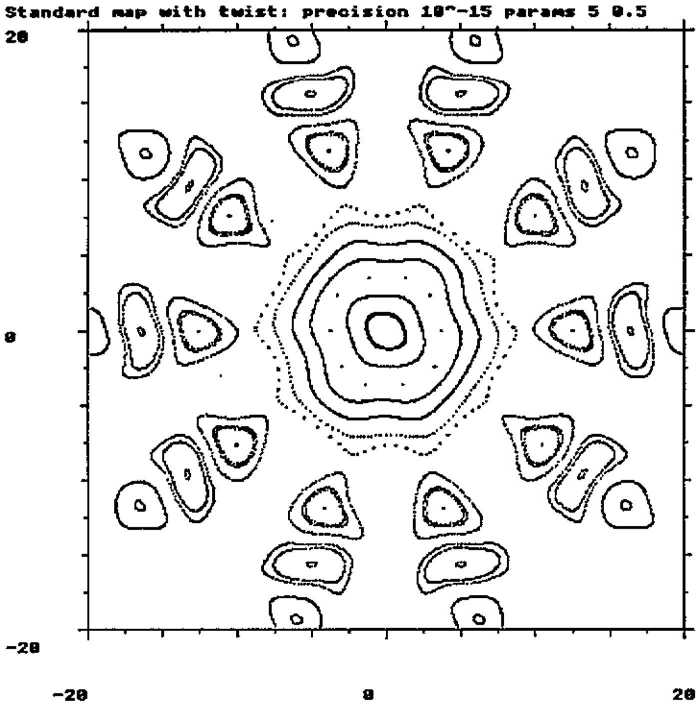

Figure 1. Trajectories of the map with twist with $n=5$, $k=0.3$, showing the (broken) intrinsic 5-fold symmetry of the map far from the origin.

periodic point: a fixed point is mapped into itself by a single application of the mapping.) In a two-dimensional map the existence of periodic points is shown by trajectories which appear to form islands around elliptic periodic points and by trajectories with sharp bends which occur near hyperbolic periodic points. One of the advantages of studying maps with only one or two variables is that it is possible to show graphically the result of iterating the map many times. (In many cases such graphical iterations produce striking images.)

When the nonlinear parameter k is zero, the map with twist displays n-fold rotational symmetry: there is a single (elliptic) fixed point at the origin. For integer values of n, every other point in the plane is mapped into itself after n applications of the mapping, i.e. every point in the plane is a periodic point of period n when $k=0$. (A similar result holds when n is a vulgar fraction.) This result is equivalent to the well known result that charged particles in a uniform magnetic field move in circular orbits, and that the frequency of rotation does not depend upon the velocity. At resonance, the map with twist is equivalent to strobing the motion of the charged particles at a multiple of the cyclotron frequency, in which case after n flashes of the strobe, a particle will have returned to its original position.

It can be shown by studying the associated Hamiltonian function (Borcherds and McCauley 1991), that there are discrete sets of points in the plane which remain as n-fold periodic points for finite values of the coefficient k. These discrete n-fold periodic points are the critical points of the hamiltonian function and are immobile in that their position does not depend on the value of k. We shall not further explicitly consider these immobile periodic points.

In addition to the immobile points of period n there are also, for all values of k greater than zero, other sets of discrete periodic points, whose period is in general not equal to n. The position of such points depends upon the value of k. In particular, as k is increased, multiple sets of points are created at the origin: as k is further increased, such sets of points move outwards from the origin, as shown in figure 2.

In this paper we study the conditions for creating such a multiple set of points at the origin, in a 'multifurcation', that is the simultaneous production of several related periodic points, and also study how the position of such points changes as k increases.

For very small values of k we shall show that there is a near universal relationship relating the position of such periodic points to the value of k at which the points were created at the origin. This relationship implies that for even the smallest finite value of k there are sets of periodic points with period other than n at all distances from the origin.

1.1. The general map with twist

Some of the results we shall be discussing apply to other maps as well as to the map with twist defined in equation (1). We define the general map with twist $\boldsymbol{G}_n$ by the equations:

$$\mathrm{G}_n(k): \quad \begin{aligned} x_{j+1} &= (x_j + kG(y_j))\cos(2\pi/n) + y_j \sin(2\pi/n) \\ x_{j+1} &= -(x_j + kG(y_j))\sin(2\pi/n) + y_j \cos(2\pi/n). \end{aligned} \tag{2}$$

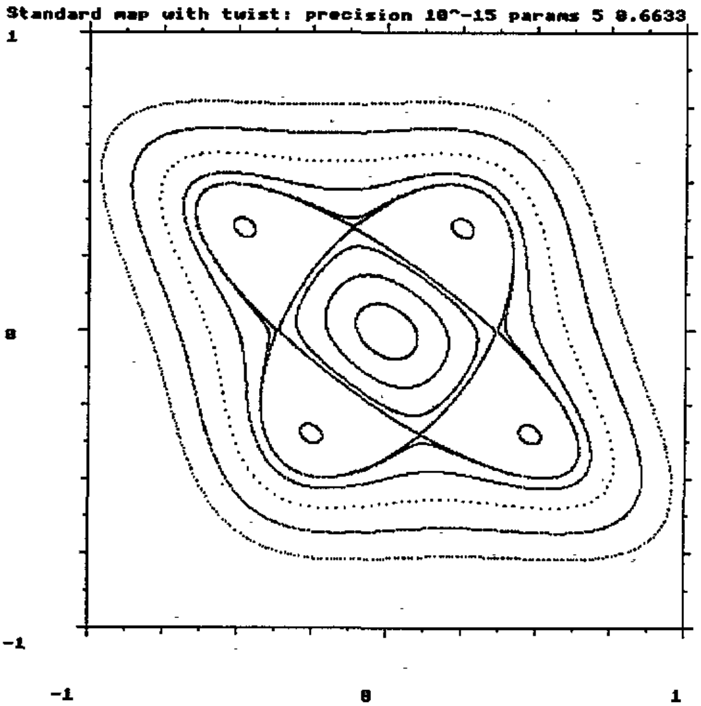

Figure 2. Trajectories of the map with twist with $n=5$, $k=0.6633$, showing the period-4 elliptic and hyperbolic periodic points near the origin. These points are born at the origin when $k=0.6498$.

where $G(y)$ is a function which can be represented by a Taylor series expansion about the origin. Our discussion will be limited to functions $G(y)$ whose Taylor series expansion contains only odd powers of y, so that the general maps we consider, like the map with twist, include only odd powers of x and y, and have odd parity.

Hénon (1969) studied the properties of *quadratic* area preserving maps. He derived very general relations for the generation of fixed and periodic points as a parameter increases. The maps we consider here do not have an explicit quadratic term, which leads to qualitative differences in the behaviour at the origin. However as the parameter k increases, we have observed that the multifurcated points we are studying do themselves eventually lose stability, and when so doing, they appear to follow the general route found by Hénon.

2. Rate of rotation at the origin

When $k=0$ the map with twist, $\mathbf{M}_n$ gives a uniform rotation about the origin, through an angle $2\pi/n$, and the associated Hamiltonian function has this symmetry too. When k is small, the trajectories of the map follow the contours of the Hamiltonian function closely. However as k increases the symmetry of the map near the origin changes, as can be seen in figure 2, in which $n=5$, a value which is not apparent from a visual inspection of the figure.

As k increases the rate of rotation about the fixed point at the origin increases steadily, but at large distances from the origin the mapping maintains an overall n-fold rotation *broken* symmetry: the trajectories trace out n paths (or $2n$ paths, if n is odd) which look like caricatures of each other rather than being identical rotated images, as can be seen in figure 1.

The rate of rotation at the origin may be determined by calculating the trace of the linearized form of the map, obtained from equation (1) by replacing $\sin(y)$ by y. For the map $\mathbf{M}_n(k)$ the trace at the origin is

$$\mathrm{trace}(\mathbf{M}_n(k)) = 2\cos(2\pi/n) - k\sin(2\pi/n). \tag{3}$$

Thus the angle of rotation at the origin is $(2\pi/m)$ where we define the *effective rotation number* m at the origin from

$$2\cos(2\pi/m) = 2\cos(2\pi/n) - k\sin(2\pi/n). \tag{4}$$

Greene *et al* (1981), considering rotations about a fixed point, define a quantity which they call the *residue* R:

$$R = (2 - \mathrm{trace}(\mathbf{M}))/4. \tag{5}$$

When the residue lies between 0 and 1 (or the trace lies between -2 and 2), the point is an elliptic point, and the (mean) rotation angle is $(2\pi/m)$ given by

$$\sin^2(2\pi/m) = R. \tag{6}$$

For other values of the residue, a fixed point is hyperbolic.

As k increases, the rotation number at the origin, m decreases, i.e.the rate of rotation at the origin increases, as observed.

We define $k_{n,m}$ to be the value of k for which the $\mathbf{M}_n$ has rotation number m at the origin. From equation (4) we find

$$k_{n,m} = 2(\cos(2\pi/n) - \cos(2\pi/m))/\sin(2\pi/n). \tag{7}$$

The behaviour of the mapping close to the origin is particularly interesting when m is either an integer or when $m=p/q$, a rational number whose denominator q is a small integer. We assume that p and q are integers with no common factor. When $k=k_{n,p/q}$ (see equation (7)), there is a p-fold multifurcation at the origin, in which two or four sets of p-fold periodic points are created. When p is even there is a single set of p elliptic points, while if p is odd there are two interleaved sets of p elliptic points. Between each pair of elliptic points is a hyperbolic point.

In the interests of brevity and clarity we shall generally refer to the rotation number associated with the number of islands as m, and only explicitly refer to p/q when it is essential to do so. The symmetry of the map is such that periodic points always occur in degenerate pairs, lying on opposite sides of the origin. We shall implicitly only consider one point of such a degenerate pair, again in the interests of brevity and clarity.

As k increases above the critical value $k_{n,p/q}$ at which the rotation number at the origin is p/q, it is found that the sets of p-fold periodic points move outwards from the origin, as described in the next section.

The phase trajectories associated with the hyperbolic points are called separatrices. There are pairs of separatrices between a hyperbolic point and each of its immediate neighbouring hyperbolic points: each pair of separatrices defines an island around the interleaved elliptic point. The islands associated with a ring of elliptic points appear rather like beads on a string, and are referred to as a necklace (of islands). As k increases the necklace moves away from the origin: each periodic point moves outwards from the origin.

In most cases the pair of separatrices defining an island, joining two hyperbolic lie close together: the island they enclose is very narrow compared with its length: indeed in most cases it is extremely difficult by visual inspection to identify the separatrices as two trajectories. Only for $m=4$ are the islands 'fat', as shown in figure 2.

2.1. $m=2$

The elliptic point at the origin loses stability in a saddle node bifurcation when

$$k=k_{n,2}=2/\tan(\pi/n) \tag{8}$$

(see equation (7)). At this value of k a pair of elliptic points is born, while the origin becomes a hyperbolic point. The range of values of the rotation number m we shall consider is $n \geqslant m \geqslant 2$.

3. The growth of the necklaces

As k increases above $k_{n,p/q}$ the sets of periodic points move outwards from the origin. We shall refer to the distance between a periodic point and the origin as the *radius* of the point.

In all the cases we have examined (except $m=2$) there are found to be periodic points on four special lines passing through the origin; namely (i) a line making an angle $(-\pi/n)$ with the x-axis, (ii) a line at right angles to (i), (iii) the x-axis and (iv) a line making an angle $(-2\pi/n)$ with the x-axis. The line with slope $(-\pi/n)$ is a symmetry line of the map.

The four sets of periodic points have coordinates and radii:

Type	Coordinates	Radius	Equation
(i)	$(-y_0/\tan(\pi/n), y_0)$	$y_0/\sin(\pi/n)$	(9*a*)
(ii)	$(y_0\tan(\pi/n), y_0)$	$y_0/\cos(\pi/n)$	(9*b*)
(iii)	$(x_0, 0)$	x_0	(9*c*)
(iv)	$(x_0\cos(2\pi/n)), -x_0\sin(2\pi/n))$	x_0	(9*d*)

This notation is appropriate for these twist maps, since we shall find, at least in some cases, formulae for x_0 or y_0, and moreover, in some cases the values of the y-coordinates for periodic points of type (i) and type (ii) are equal.

It can easily be seen, from equation (1) that a point of type (iii) maps into a point of type (iv) lying on the line passing through the origin, with slope $(-2\pi/n)$. Thus type (iv) points are trivially related to type (iii) points.

An initial numerical exploration of the map suggested that for small values of $\Delta k(\Delta k = k - k_{n,m})$ the radius of the sth periodic point in the chain was proportional to $\surd(\Delta k/k_{n,m})$.

$$r_{k,n,m,s} \cong Q_{n,m,s}\surd(\Delta k/k_{n,m}) \tag{10}$$

where the constant of proportionality, $Q_{n,m,s}$ depends on the values of n, m and is generally not the same for all periodic points in the chain. As we shall see in the next section, equation (10) is an approximation, valid only when the higher order terms in $G(y)$ (equation (2)) may be neglected.

If there were a single equation, like equation (10), relating the radii of all the points in the chain to the parameter k, that would imply that the shape of a chain would not change as k was varied. However the shape of the chain, which depends strongly on the values of n and m, also depends weakly on the value of k. When m is very close to n (and $k_{n,m}$ is very small, see equation (7)), the periodic points lie very nearly on a circle. As the difference between n and m increases the oval on which the periodic points lie becomes increasingly elongated; with its major axis lying on the symmetry line of the map, with slope $(-\pi/n)$.

To determine analytically how the position of a periodic point with rotation number p/q depends on the value of k would involve the repeated application of the map (equation (1)) p times. Like all nonlinear maps, the map with twist becomes intractable when iterated. Even the second iteration is in general intractable: the use of computer algebra packages such as *Reduce* or *Maple* still leaves results too complicated to yield real insight. However, from the numerical results it was observed for $m=3$, 4 and 6, and for a fixed value of the ratio $k/k_{n,m}$, that several of the periodic points in the chain had the same value for the y-coordinate. This encouraged us to search for simple relations for those periodic points with this property: the relations found are described in the next section.

In the initial numerical work we examined how the radii of the periodic points depended on the values of the intrinsic rotation number, n, and the effective rotation number, m. Some results for the different types of periodic points are shown in figures 3 and 4, for $n=4$ and $n=6$ respectively, for $\Delta k/k_{n,m}=10^{-6}$. For other values of n the results are similar to figure 4.

In the limit where m approaches n (when $k_{n,m}$ approaches zero) on the right hand

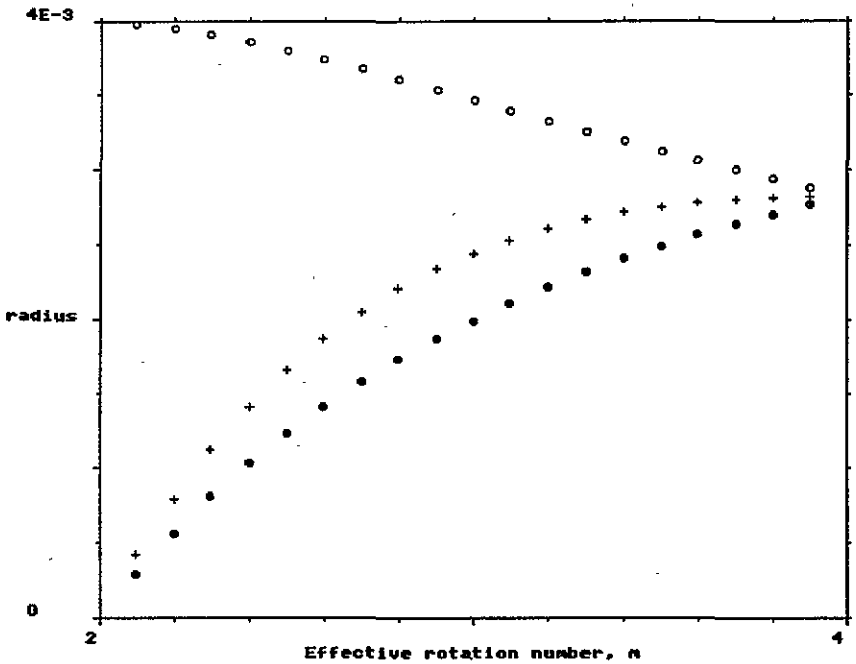

Figure 3. The 'radius' of periodic points of types (i) ○, (ii) ● and (iii) + for $n=4$ and $\Delta k/k_c=10^{-6}$.

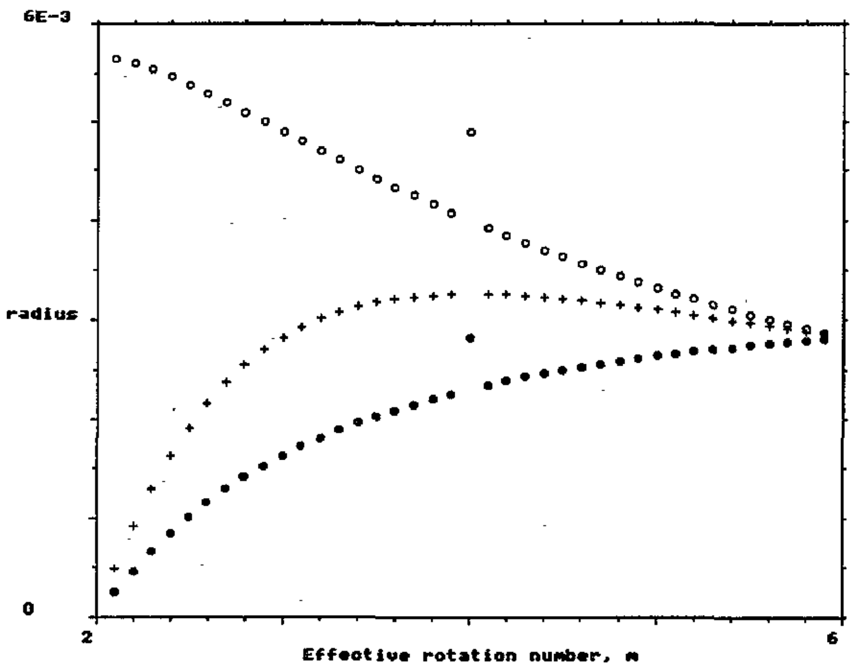

Figure 4. The 'radius' of periodic points of types (i) ○, (ii) ● and (iii) + for $n=6$ and $\Delta k/k_c=10^{-6}$: note $m=4$ is anomalous.

side of the figures, $Q_{n,m,s}$ approaches a constant value ($\sqrt{8}$) for all three types of periodic points, and the necklace of islands is circular. As m decreases, the island of necklaces associated with a set of periodic points becomes increasingly elongated: this is reflected in the separation of the upper and lower curves in these two figures. At $m=2$ there are only two type (i) periodic points, the radii of the other types of periodic points approach zero as m approaches 2. The variation of radius with m is smooth, except when $m=4$, when the results are anomalous, as can be seen in figure 4: in many of the cases we have studied, the value of $Q_{n,4,s}$ is equal to $Q_{n,3,s}$.

In initial investigations the intrinsic rotation number of the mapping, n was set equal to a small integer, and chains with rotation number m investigated. If n is an integer and $m=n-1/q$, it is necessary to carry out $(n\,q-1)$ iterations to map a periodic point onto itself. It was found impractical to make q greater than 1000: for greater values the effect of rounding errors resulted in inconsistent results (when carrying out calculations with a precision of about 10^{-9}). The investigations for very small values of k (large values of q) were considerably speeded up when it was realised that instead of making n an integer and keeping it constant while varying m, the problem could be inverted: make m a small integer (3, 4 or 6) and vary n: n takes on fractional values close to m, say $m+1/q$. Now, irrespective of the value of q it is necessary to carry out only m iterations to return to a periodic point and this allowed the system to be studied for much smaller values of $(n-m)$.

4. Some values of m: 2, 3, 4 and 6

Inspection of the values of the coordinates of the complete cycle of periodic points shows that in many cases pairs of points not too far apart in the chain have the same ordinate. This observation encouraged us to look for analytic results. In some cases we found that the pattern of numbers was no more than an identity, but in others we did manage to find, for certain values of m, results which enabled us to obtain an explicit relation between the coordinates of a periodic point and k.

The values of m for which we have been able to derive some analytic results are $m=2$, 3, 4 and 6. We note that these are the rotation numbers allowed in a periodic tiling of the plane.

The analytically tractable cases are those for which y_1, the ordinate of the *first iterate*, of a type (i) or a type (ii) periodic point (see equation (9)) is either zero or else *numerically* equal to y_0, the ordinate of the point being mapped.

4.1. *Type (i) periodic points*

Applying the *general* map with twist to a type (i) periodic point we obtain for the y-coordinate of the first iterate

$$y_1=y_0(1+2\cos\alpha)-k\,G(y_0)\sin\alpha. \tag{11}$$

Examining the numerical results we find that when $m=2$ then $y_1=-y_0$, when $m=3$ then $y_1=0$, and when $m=4$ then $y_1=y_0$. From these results we obtain relations for the value of y_0 for the three cases which can all be written

$$y_0/G(y_0)=k/k_{n,m} \tag{12}$$

where $k_{n,m}$ is the critical parameter value at which the appropriate set of periodic points is born.

4.2. *Type (ii) periodic points*

There is no period 2 type (ii) periodic point. Examining the numerical results, we find that when $m=4$ then $y_1=-y_0$, and when $m=6$ then $y_1=0$, and for both cases y_0 satisfies equation (12).

When $m=3$ we find from the numerically that $y_2=y_0$.

The general expression for the y coordinate of the second image of a type (ii) periodic point is

$$y_2=(4\cos^2\alpha-2\cos\alpha-1)y_0-k\sin\alpha\,[2\cos\alpha G(y_0)+G\{(2\cos\alpha-1)y_0-k\sin\alpha\, G(y_0)\}] \qquad (13)$$

where $\alpha=2\pi/n$.

When $m=3$

$$k_{n,m}=(2\cos\alpha+1)/\sin\alpha \qquad (14)$$

Setting $y_2=y_0$, and making the substitution for $k_{n,m}$ from equation (14), we derive, from equation (13), the relation

$$y_0=(k/k_{n,m})[2\cos\alpha\, G(v)+G\{(2\cos\alpha-1)y_0-k\sin\alpha\, G(y_0)\}]/\{2(\cos\alpha-1)\}. \qquad (15)$$

The dependence of y upon k given by equation (15) is qualitatively different from that given by equation (12). In particular we note that k appears inside the argument of G, and not only in the ratio $k/k_{n,m}$.

4.3. $m=3$

When $m=3$, the three elliptic points are of types (i), (iii) and (iv). For two of them the y-coordinate satisfies equation (12), and the radius of the type (iv) point is equal to that of the type (iii) point. The shape and orientation of the figure defined by these points is independent of k.

The position of the hyperbolic point, which is of type (ii) is determined by equation (15), thus the shape of the chain of all the periodic points does depend on k. Thus for $m=3$ we see that the radii of the type (i) (elliptic) periodic points and the type (ii) (hyperbolic) periodic points grow at different rates, thus the shape of the chain of islands changes as it grows.

4.4. $m=4$

As can be seen from figure 4, the results for $m=4$ are anomalous: it appears that it is only when m is exactly equal to 4 that there is an anomaly. We have measured the 'radius' of periodic points for $m=4.0001$ and for $m=3.9999$; for neither of these values of m are the results anomalous.

When $m=4$, the two elliptic points are of type (i) and (ii): their y-coordinates are equal, and are determined by equation (12). The two hyperbolic points are of type (iii) and (iv): their radii are equal, and the y-coordinate of the type (iv) point is also determined by equation (12). When $n=6$ it is easy to show that the periodic points of type (ii), (iii) and (iv) all have the same radius.

All the periodic points for $m=4$ lie on special lines, the shape and orientation of the chain of periodic points is independent of k. It appears from visual inspection that the islands defined by the separatrices do not change in shape as k increases. It also appears that the separatrices lie on a pair of intersecting ellipses, but this has not been verified analytically.

4.5. $m=6$

For $m=6$, the three hyperbolic points are of types (ii), (iii) and (iv). For two of them the value of the y-coordinate is given by equation (12), and the type (iv) point has the same radius as the type (ii) point, thus the shape and orientation of the figure defined by the hyperbolic points is independent of k.

The position of the type (i) point, which is elliptic is determined by an equation (similar to equation 15) for which an analytic solution is not practicable. The second image of this point has the same value of the y-coordinate as the point itself. As for $m=3$, the two sets of points do not have the same dependence on k. The shape of the chain is not constant.

4.6. *Other values of m*

For other values of m we have not succeeded in deriving any relations. For $m=5$ we observe that type (i) periodic points satisfy $y_4=-y_0$ and type (ii) points satisfy $y_4=y_0$. For $m=8$, type (i) periodic points and type (ii) periodic points belong to the same cycle and satisfy $y_3=y_0$. For the type (iii) periodic points we observe $y_3=y_1$. None of these results will yield expressions simple enough to be worth pursuing analytically, but it is clear from them, and from the discussion of equation (15) that the *shapes* of the chains of periodic points do depend on the value of k.

5. A universal limit as $m \to n$ ($k \to 0$)

The numerical results show that the necklace of islands is elongated along the symmetry line of the map, and that the 'eccentricity' of the necklace increases as $(n-m)$ increases. As $(n-m)$ decreases to zero the necklace becomes increasingly circular, and as the radii tend to zero, they all tend to the same value, both for elliptic and hyperbolic points (except when $m=4$).

When $m=3$ the y-coordinate of the (elliptic) periodic point of type (i) is given by equation (12): this periodic point has coordinates $(-y_0/\tan(\pi/n), y_0)$, and its radius is $y_0/\sin(\pi/n)$.

The other two period-3 elliptic periodic points have coordinates $(y_0/\sin(\pi/n), 0)$ and $(y_0/\tan(2\pi/n), -y_0)$, and they both have radius $y_0/\sin(2\pi/n)$.

To obtain an explicit formula for the radius in the limit $k \to 0$, we set $G(y_0)=\sin(y_0)$ in equation (12) and expand it as a Taylor series:

$$1+y^2/6+O(y^4)=1+\Delta k/k_{n,m} \tag{16}$$

or

$$y_0=\sqrt{(6\Delta k/k_{n,m})}. \tag{17}$$

In the limit $n \to 3$ we obtain for the radius, r for all three elliptic points, since $\sin(\pi/3)=\sin(\pi/6)$

$$r=\sqrt{(8\Delta k/k_{n,m})}. \tag{18}$$

For all other values of m except $m=4$ the radius as calculated numerically agrees with equation (18) in the limit $n-m\rightarrow 0$, but this ($m=3$) is the only case for which an analytic result has been obtained.

We note from figure (4) that $m=4$ is anomalous: the type (i) elliptic points, on the symmetry axis satisfy equation (18), but the radii of the hyperbolic points are smaller by a factor $\sqrt{2}$.

6. Discussion

By using a combination of numerical and analytic techniques to investigate the positions of the multifurcated m-fold periodic points of the map with twist which are created at the origin, we have been able to obtain a universal result (equation (18)) for how their radii depend on the nonlinear parameter k, when k is small. This is illustrated in figures 3 and 4, where all the curves tend to the same point for $k=0$, at the right-hand side of the figure.

A general direct analytic approach is impractical, as the iterated expressions of the map are intractable, however, from numerical results, some simple relations involving only a single iteration of the mapping were observed for certain values of the periodicity m of the points, and these relations have enabled us to identify cases susceptible to an analytic treatment.

When m, the rotation number of the points in the chain, is close in value to n, the intrinsic rotation number of the map, it is convenient to write $m=n-1/q$. Then the value of $k_{n,m}$, in the limit of large q is

$$k_{n,m}=2\sin(2\pi/q)=4\pi/q=4\pi(n-m). \tag{19}$$

Combining equations (17) and (19), we see in the limit as $(n-m)\rightarrow 0$, that for even the smallest value of k greater than zero, there are sets of multifurcated periodic points, which were created at the origin, at large distances from the origin. How far such a set of multifurcated points is from the origin depends on the ratio $\Delta k/k_{n,m}$. The smaller the value of $k_{n,m}$, the further, for a given value of Δk are the points from the origin.

The rate at which a periodic point moves away from the origin as k increases depends upon the ratio $\Delta k/k_{n,m}$, the smaller the value of $k_{n,m}$ the faster the rate at which periodic points move away. When k is tiny, then q is large, and so too is the number of periodic points, namely $(q\,n-1)$.

No matter how small k is, there are periodic points at all distances, whose distance from the origin increases rapidly as k increases. As $k\rightarrow 0$, the number of periodic points in a chain increases without limit: the chain of periodic points becomes increasingly difficult to distinguish from a continuous curve, but nevertheless is comprised of alternating elliptic and hyperbolic points. In the limit, when $k=0$, then *all* the points in the plane are periodic points (of period n), and the distinction between elliptic and hyperbolic points ceases to have any meaning: the sole exception is the origin itself, which is a *fixed* point (period 1).

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